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On the Asymptotic Behavior of a Minimal Surface over a Semi-infinite Strip

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This paper is concerned with the asymptotic behavior of solutions of the minimal surface equation in two dimensions, which vanish on the long sides of a semi-infinite strip. It is shown that as the axial coordinate x_1 tends to infinity, such solutions of the minimal surface equation with nonnegative Dirichlet data on the end $x_1 = 0$, *decay exponentially* in x_1 and *at precisely the same rate* as do harmonic functions. The results are relevant to principles of Saint–Venant and Phragmén–Lindelöf type. © 1990 Academic Press, Inc.

1. INTRODUCTION

Some time ago, Knowles [1] established a spatial decay estimate for solutions $u(x_1, x_2)$ of the minimal surface equation

$$L[u] = (1 + u_{,2}^2)\, u_{,11} - 2u_{,1} u_{,2} u_{,12} + (1 + u_{,1}^2)\, u_{,22} = 0 \tag{1}$$

over the semi-infinite strip $0 \leqslant x_1 < \infty$, $0 \leqslant x_2 \leqslant h$. It was shown in [1] that solutions of (1), which are continuously differentiable on the closed semi-infinite strip, twice continuously differentiable on its interior, and satisfy the boundary conditions

$$u(x_1, 0) = u(x_1, h) = 0, \qquad 0 \leqslant x_1 < \infty, \tag{2}$$

$$u(x_1, x_2) \to 0 \quad \text{as} \quad x_1 \to \infty \text{ (uniformly in } x_2), \qquad 0 \leqslant x_2 \leqslant h, \tag{3}$$

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decay exponentially in x_1 *at least as fast as do solutions of Laplace's equation subject to the same boundary conditions.*

The explicit decay estimate establishing this result is given in [1] as

$$|u(x_1, x_2)| \leqslant M \sin(\pi x_2/h) \exp(-\pi x_1/h), \qquad x_1 \geqslant 0, \qquad 0 \leqslant x_2 \leqslant h, \quad (4)$$

where

$$M = \sup_{0 < x_2 < h} \left\{ \frac{|u(0, x_2)|}{\sin(\pi x_2/h)} \right\} \geqslant 0; \quad (5)$$

M is finite since $u(0, x_2)$ is continuously differentiable for $0 \leqslant x_2 \leqslant h$ and vanishes at $x_2 = 0$ and $x_2 = h$. It is easily shown that the foregoing result also holds for solutions of Laplace's equation satisfying (2), (3); in this case the decay rate π/h is the *best possible* (see, e.g., [1, 2, 10]).

In this paper, stronger results are established. First we show that the result (4), (5) holds for solutions of the minimal surface equation (1) *which merely satisfy conditions* (2) *only*. Thus it will be established that the hypothesis (3), namely that u vanishes (uniformly in x_2) as $x_1 \to \infty$, is *not necessary* in order to conclude (4), (5). In fact, this result stated in Theorem 1 below, *shows that solutions of* (1) *which satisfy* (2), *automatically satisfy the asymptotic condition* (3). Second, we show that the decay rate π/h predicted by the estimate (4) is the *best possible* for the minimal surface equation. This is established by combining the *upper bound* (4), (5) with the *lower bound* result (21) stated in Theorem 2 for positive solutions of (1), (2). Thus we conclude that *for sufficiently large values of* x_1, *positive solutions to the minimal surface equation* (1) *satisfying the boundary conditions* (2) *decay exponentially in* x_1 *at the same rate as do harmonic functions satisfying* (2), (3). Finally we show (see Theorem 3) that a stronger result than that of Theorem 1 may be established where the boundary conditions (2) are assumed to hold merely in a limiting sense as $x_1 \to \infty$.

2. First Result

THEOREM 1. *Let* $u(x_1, x_2)$ *be a twice continuously differentiable solution of the minimal surface equation* (1) *over the semi-infinite strip* $0 < x_1 < \infty$, $0 < x_2 < h$, *which is continuous on the closed semi-infinite strip. Suppose that*

$$u(x_1, 0) = u(x_1, h) = 0 \qquad \textit{for} \quad x_1 \geqslant 0. \quad (6)$$

Then

$$\lim_{x_1 \to \infty} u(x_1, x_2) = 0 \qquad (\textit{uniformly in } x_2) \qquad \textit{for} \quad 0 \leqslant x_2 \leqslant h. \quad (7)$$

Proof. The idea of the proof is to obtain an upper bound for solutions u of (1), (2) in terms of a known explicit solution of (1) describing a catenoid surface. For this purpose, it is convenient to consider Eq. (1) over the semi-infinite strip $0 < x_1 < \infty$, $-h < x_2 < h$ (symmetric with respect to x_1-axis) and thus, instead of (6), we have

$$u(x_1, \pm h) = 0, \qquad x_1 \geqslant 0. \tag{8}$$

Consider the catenoid surface generated by rotating the catenary curve

$$z = a\left[\cosh\left(\frac{h}{a}\right) - \cosh\left(\frac{x_2}{a}\right)\right] \qquad (a > 0), \tag{9}$$

about the axis

$$z = a\cosh\left(\frac{h}{a}\right), \qquad x_1 = a\cosh\left(\frac{h}{a}\right). \tag{10}$$

The lower half of the catenoid surface thus generated is given by

$$z = v(x_1, x_2) = a\cosh\left(\frac{h}{a}\right) - \left\{a^2\cosh^2\left(\frac{x_2}{a}\right) - \left[x_1 - a\cosh\left(\frac{h}{a}\right)\right]^2\right\}^{1/2}, \tag{11}$$

where $a > 0$ is an arbitrary constant. It may be verified directly that $L[v] = 0$ so that v is a solution to the minimal surface equation (1). We denote by Ω the projection of the surface $z = v(x_1, x_2)$ onto the $x_1 - x_2$ plane. Thus the domain Ω, shown in Fig. 1, is bounded by the horizontal lines Γ_1, Γ_2 and the two catenaries Γ_3, Γ_4. We now show that

$$u \leqslant v \qquad \text{on } \Omega. \tag{12}$$

To establish (12) we use an argument based on ideas of R. Finn [3, pp.

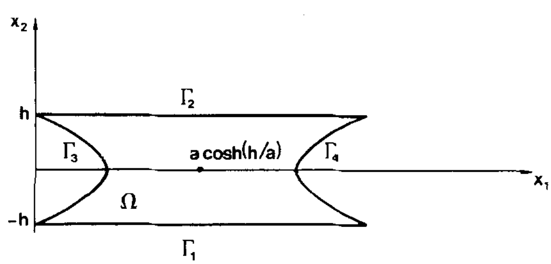

FIG. 1. The comparison domain Ω used in the proof of Theorem 1.

139–140]. The same sort of argument can be traced back to S. N. Bernstein [4].

On using (8) and the definition of v in (11), it may be easily verified that

$$u \leqslant v \qquad \text{on } \Gamma_1 \text{ and } \Gamma_2. \tag{13}$$

Since the vertical cross-sections of the catenoid surface in the x_1-direction are lower half-circles, it follows that if $q^* = (x_1^*, x_2^*)$, $|x_2^*| < h$,

$$\lim_{x_1 \to x_1^*+} \frac{\partial v}{\partial x_1}(x_1, x_2^*) = -\infty, \qquad \text{if} \quad q^* \in \Gamma_3 \tag{14}$$

$$\lim_{x_1 \to x_1^*-} \frac{\partial v}{\partial x_1}(x_1, x_2^*) = \infty, \qquad \text{if} \quad q^* \in \Gamma_4. \tag{15}$$

Let $w = v - u$. It is well known [5, 6] that w satisfies a homogeneous linear elliptic equation with no zero order term. Denote by m the minimum value of w on $\bar{\Omega}$. By the strong maximum principle, this minimum value must occur at some point q on the boundary of Ω. We next argue that q cannot be an interior point of Γ_3 or Γ_4. If $q \in \operatorname{int} \Gamma_3$, then $q = (x_1^*, x_2^*)$, $|x_2^*| < h$, $w(q) = m$, and $\lim_{x_1 \to x_1^*+} (\partial w / \partial x_1)(x_1, x_2^*) = -\infty$ by (14). It follows that $w(x_1, x_2^*) < m$ for values of x_1 close to x_1^*; this contradicts the definition of m. The argument for $q \in \operatorname{int} \Gamma_4$ is analogous and makes use of (15). The only remaining possibility is that $q \in \Gamma_1$ or $q \in \Gamma_2$. By (13), $w \geqslant 0$ on Γ_1 and Γ_2, hence $m \geqslant 0$ (in fact $m = 0$). Thus we have established that $w \geqslant 0$ on $\bar{\Omega}$ which implies, in particular, that (12) holds.

By virtue of (12), (11) we have

$$u\left(a \cosh \frac{h}{a}, x_2\right) \leqslant a \cosh\left(\frac{h}{a}\right) - a \cosh\left(\frac{x_2}{a}\right) \leqslant a \cosh\left(\frac{h}{a}\right) - a. \tag{16}$$

On replacing u by $-u$ in the preceding analysis, we may also conclude that $-u \leqslant v$ on Ω and so find that

$$-u\left(a \cosh \frac{h}{a}, x_2\right) \leqslant a \cosh\left(\frac{h}{a}\right) - a. \tag{17}$$

Thus (16), (17) yield

$$\left|u\left(a \cosh \frac{h}{a}, x_2\right)\right| \leqslant a\left(\cosh \frac{h}{a} - 1\right), \tag{18}$$

an estimate that is independent of x_2. On letting $a \to \infty$ we see that $a\cosh(h/a) \to \infty$ and the right hand side of (18) tends to zero and so

$$\lim_{x_1 \to \infty} u(x_1, x_2) = 0 \qquad (\text{uniformly in } x_2), \tag{19}$$

which is the desired conclusion. This completes the proof of Theorem 1.

3. Second Result

The second result that we establish may be stated as follows:

THEOREM 2. *Let $u(x_1, x_2)$ be a twice continuously differentiable solution to the minimal surface equation* (1) *over the semi-infinite strip $0 < x_1 < \infty$, $0 < x_2 < h$, which is continuous on the closed strip, and satisfies the boundary conditions $u(x, 0) = u(x_1, h) = 0$, $x_1 \geqslant 0$. Suppose that*

$$u(0, x_2) \geqslant 0, \qquad u(0, x_2) \not\equiv 0 \qquad \textit{for} \quad 0 \leqslant x_2 \leqslant h. \tag{20}$$

Then

$$u(x_1, x_2) \geqslant c \sin(\pi x_2/h) \exp(-\pi x_1/h), \qquad 0 \leqslant x_2 \leqslant h, \tag{21}$$

for $x_1 \geqslant (n/\pi)h$, for some positive constant c, $0 < c \leqslant 1$, where n is the positive zero of a transcendental expression defined in (37), $n \doteq 0.7738$.

Proof. The proof of Theorem 2 again makes use of a comparison principle for solutions of quasilinear elliptic partial differential equations. It is convenient to introduce the dimensionless variables

$$x = \frac{\pi x_1}{h}, \qquad y = \frac{\pi x_2}{h}, \tag{22}$$

and the notation

$$R_{x^0, x^1} = \{(x, y) \mid x^0 < x < x^1, 0 < y < \pi\}. \tag{23}$$

Thus $u(x, y)$, which we write for $(\pi/h)\, u(x_1, x_2)$, satisfies the minimal surface equation in the (x, y) variables on the semi-infinite strip $R_{0,\infty}$ and the boundary conditions

$$u(x, 0) = u(x, \pi) = 0, \qquad x \geqslant 0. \tag{24}$$

By virtue of Theorem 1, we know that

$$u(x, y) \to 0 \qquad (\text{uniformly in } y) \qquad \text{as} \quad x \to \infty. \tag{25}$$

We first show that

$$u(x, y) > 0 \qquad \text{on } R_{0,\infty}. \tag{26}$$

Let (x^0, y^0) be an arbitrary point in $R_{0,\infty}$ and let $x^1 > x^0$. On R_{0,x^1}, u attains its minimum on the boundary by virtue of the maximum principle (see, e.g., [5] or [6]) and so

$$u(x^0, y^0) \geqslant \min\{\min_{x=0} u, \min_{x=x^1} u, 0\}. \tag{27}$$

On letting $x^1 \to \infty$, we deduce with the aid of (25) that

$$u(x^0, y^0) \geqslant \min\{\min_{x=0} u, 0\} \geqslant 0, \tag{28}$$

the final inequality being true by virtue of (20). Thus $u(x, y) \geqslant 0$ on $R_{0,\infty}$. By the strong maximum principle [5, 6] u cannot attain its minimum value, namely zero, at an interior point (since $u \not\equiv 0$) and so (26) follows.

We now show that

$$u \geqslant v \qquad \text{on } R_{x^0,\infty}, \tag{29}$$

where $x^0 = n$ and

$$v = c\left(1 + \frac{1}{x}\right) e^{-x} \sin y, \tag{30}$$

where c is a constant, $0 < c \leqslant 1$, to be chosen later. Denoting the minimal surface operator in the (x, y) variables by $L[v]$ (see (1)), a direct calculation using (30) shows that

$$L[v]/[e^{-x} \sin y] = 2cx^{-2}(1 + x^{-1}) + c^3 F(x, y)\, e^{-2x}, \tag{31}$$

where

$$F(x, y) = (1 + x^{-1})[(2x^{-3} + x^{-4}) \cos^2 y - (1 + x^{-1} + x^{-2})^2]. \tag{32}$$

Now

$$F(x, y) \geqslant -(1 + x^{-1})(1 + x^{-1} + x^{-2})^2 \qquad \text{on } R_{0,\infty} \tag{33}$$

and so, from (31), we deduce that

$$L[v]/[c(1 + x^{-1})\, e^{-x} \sin y] \geqslant 2x^{-2} - c^2(1 + x^{-1} + x^{-2})^2\, e^{-2x} \qquad \text{on } R_{0,\infty}. \tag{34}$$

It will be shown below that the constant c can be chosen such that

$$0 < c \leqslant 1. \tag{35}$$

Using the right hand side inequality of (35) in (34), we deduce that

$$L[v] \geqslant 0 \qquad \text{on } R_{x^0, \infty}, \tag{36}$$

provided a value of x^0 can be determined such that

$$f(x) \equiv 2x^{-2} - (1 + x^{-1} + x^{-2})^2 e^{-2x} \geqslant 0 \qquad \text{for} \quad x \geqslant x^0. \tag{37}$$

It may be readily verified that the function $f(x)$ has a *single* zero on $(0, \infty)$, namely at

$$x^0 = n \doteq 0.7738, \tag{38}$$

and that $f(x) > 0$ for $x > n$. Thus (37), and so (36), has been established for the value $x^0 = n$ given in (38).

The constant c appearing in (30) is now chosen as follows. Let

$$c = \min\{1, q\}, \tag{39}$$

where

$$q = \inf_{0 < y < \pi} \left\{ \frac{u(x^0, y)}{e^{-x^0}(1 + 1/x^0) \sin y} \right\} \tag{40}$$

$$= \frac{1}{e^{-x^0}(1 + 1/x^0)} \inf_{0 < y < \pi} \left\{ \frac{u(x^0, y)}{\sin y} \right\}. \tag{41}$$

We observe that the function

$$g(y) \equiv u(x^0, y)/\sin y \tag{42}$$

is such that

$$g(y) > 0 \qquad \text{on} \quad 0 < y < \pi, \tag{43}$$

by virtue of (26). Furthermore,

$$\liminf_{y \to \pi-} \frac{u(x^0, y)}{\sin y} = \liminf_{y \to \pi-} \frac{u(x^0, y)}{\pi - y} > 0, \tag{44}$$

the final inequality following as a consequence of Hopf's boundary point lemma (see [5, p. 67] or [6, p. 33]). A similar argument shows that

$$\liminf_{y \to 0+} \frac{u(x^0, y)}{\sin y} > 0. \tag{45}$$

Consequently the constant q defined in (41) is positive and so c defined by (39) is such that $0 < c \leqslant 1$. Thus (35) has been established.

To reach the desired conclusion (29), we finally apply a comparison principle [5, 6] for the minimal surface equation on R_{x^0, x^1}, where x^1 is a sufficiently large value of x, $x^1 > x^0$. Introduce $v_\varepsilon = v - \varepsilon$ for $\varepsilon > 0$. Since $L[v_\varepsilon] = L[v]$ we have by (36) that

$$L[v_\varepsilon] = L[v] \geqslant 0 = L[u] \qquad \text{on } R_{x^0, x^1}. \tag{46}$$

Also by virtue of (24) and (30)

$$v_\varepsilon(x, 0) < v(x, 0) = u(x, 0); \qquad v_\varepsilon(x, \pi) < v(x, \pi) = u(x, \pi), \qquad x^0 \leqslant x \leqslant x^1, \tag{47}$$

whereas, by (30), (39), (40),

$$v_\varepsilon(x^0, y) < v(x^0, y) \leqslant u(x^0, y). \tag{48}$$

Furthermore (26), (30) ensure that for x^1 chosen sufficiently large,

$$v_\varepsilon(x^1, y) < u(x^1, y), \qquad 0 \leqslant y \leqslant \pi. \tag{49}$$

It follows from the maximum principle that

$$u \geqslant v_\varepsilon \qquad \text{on } R_{x^0, x^1}, \tag{50}$$

and so on letting $x^1 \to \infty$ and $\varepsilon \to 0$, we reach the desired result (29). Since $c > 0$, we deduce from (29), (30) that

$$u \geqslant ce^{-x} \sin y \qquad \text{on } R_{x^0, \infty}, \tag{51}$$

where c is given by (39), (40). The result (21) in the statement of Theorem 2 now follows from (51) on returning to the original (x_1, x_2) coordinates on using Eq. (22). This completes the proof of Theorem 2.

We remark that, by a translation of the comparison function used in the preceding proof, the conclusion of Theorem 2 can be seen to hold for $x_1 \geqslant k$, for any positive k.

4. Concluding Remarks

It should be noted that Theorem 1 here also follows immediately from a result established by Nitsche in 1975 (see [7, p. 709]). In fact, under the hypotheses of Theorem 1, Nitsche has shown that, for sufficiently large x_1, $u(x_1, x_2)$ *decays exponentially* with x_1 (uniformly in x_2) from which the conclusion (7) immediately follows. However, the exponential decay *rate* obtained in [7] is not as sharp as that of (4). The arguments used in [7] are based on a maximum principle for the minimal surface equation (see [8, p. 256]) as well as comparison with a particular minimal surface. A less explicit decay estimate for solutions of the minimal surface equation on a semi-infinite strip was obtained earlier by Roseman [9].

Finally, we note that a stronger result than that of Theorem 1 may be established by modifying the construction in the proof of that theorem. We show that the conclusion (7) of Theorem 1 may still be arrived at when the hypothesis (6) is assumed to hold merely in a limiting sense as $x_1 \to \infty$. This result may be stated as follows:

THEOREM 3. *Let $u(x_1, x_2)$ be a twice continuously differentiable solution to the minimal surface equation* (1) *over the semi-infinite strip $0 < x_1 < \infty$, $0 < x_2 < h$, which is continuous on the closed semi-infinite strip. Suppose that*

$$\lim_{x_1 \to \infty} u(x_1, 0) = \lim_{x_1 \to \infty} u(x_1, h) = 0. \tag{52}$$

Then

$$\lim_{x_1 \to \infty} u(x_1, x_2) = 0 \qquad (\textit{uniformly in } x_2) \qquad \textit{for} \quad 0 \leqslant x_2 \leqslant h. \tag{53}$$

Proof. As in the proof of Theorem 1, we again work over the symmetric semi-infinite strip $0 < x_1 < \infty$, $-h < x_2 < h$, and thus instead of (52) we have

$$\lim_{x_1 \to \infty} u(x_1, -h) = \lim_{x_1 \to \infty} u(x_1, h) = 0. \tag{54}$$

Now let $g(a) = \max\{\sup_{s \geqslant a} |u(s, -h)|, \sup_{s \geqslant a} |u(s, h)|\}$, where $a > 0$ is an arbitrary constant. Define $\hat{v}(x_1, x_2) = v(x_1 - a, x_2) + g(a)$, where the function v is defined as in Eq. (11). By arguing as before, we have $u \leqslant \hat{v}$, from which follows

$$u\left(a \cosh \frac{h}{a} + a, x_2\right) \leqslant a\left(\cosh \frac{h}{a} - 1\right) + g(a). \tag{55}$$

This same inequality holds with u replaced by $-u$, hence

$$\left|u\left(a\cosh\frac{h}{a}+a, x_2\right)\right| \leqslant a\left(\cosh\frac{h}{a}-1\right)+g(a). \tag{56}$$

On letting $a \to \infty$, we see from (54) and the definition of $g(a)$ that the right hand side of (56) tends to zero, and so we obtain the desired conclusion

$$\lim_{x_1 \to \infty} u(x_1, x_2) = 0 \qquad \text{(uniformly in } x_2\text{)}. \tag{57}$$

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REFERENCES

1. J. K. KNOWLES, A note on the spatial decay of a minimal surface over a semi-infinite strip, *J. Math. Anal. Appl.* **59** (1977), 29–32.
2. C. O. HORGAN AND J. K. KNOWLES, Recent developments concerning Saint-Venant's principle, *in* "Advances in Applied Mechanics" (T. Y. Wu and J. W. Hutchinson, Eds.), Vol. 23, pp. 179–269, Academic Press, New York, 1983.
3. R. FINN, Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature, *J. Analyse Math.* **14** (1965), 139–160.
4. S. N. BERNSTEIN, Sur les surfaces définies au moyen de leur courbure moyenne ou totale, *Ann. Sci. Ecole Norm. Sup.* **27** (1909), 233–256.
5. M. H. PROTTER AND H. F. WEINBERGER, "Maximum Principles in Differential Equations," Springer-Verlag, Berlin, 1984.
6. D. GILBARG AND N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, Berlin, 1977.
7. J. C. C. NITSCHE, "Vorlesungen über Minimalflächen," Springer-Verlag, Berlin, 1975.
8. J. C. C. NITSCHE, On new results in the theory of minimal surfaces, *Bull. Amer. Math. Soc.* **71** (1965), 195–270.
9. J. J. ROSEMAN, The rate of decay of a minimal surface defined over a semi-infinite strip, *J. Math. Anal. Appl.* **46** (1974), 545–554.
10. C. O. HORGAN, Recent developments concerning Saint-Venant's principle: An update, *Appl. Mech. Rev.* **42** (1989), 295–303.